Temperature Fluctuations and Entropy Formulas

Does it or does not?

T.S. Biró G.G. Barnaföldi P. Ván

Heavy Ion Research Group
MTA Wigner Research Centre for Physics, Budapest

February 9, 2014

J.Uffink, J.van Lith: Thermodynamic Uncertainty Relations; Found.Phys.29(1999)655

”Bohr and Heisenberg suggested that the thermodynamical fluctuation of temperature and energy are complementary in the same way as position and momenta in quantum mechanics.”

B.H.Lavenda: Comments on "Thermodynamic Uncertainty Relations" by J.Uffink and J.van Lith; Found.Phys.Lett.13(2000)487

”Finally, the question about whether or not the temperature really fluctuates should be addressed. ... If the energy fluctuates so too will any function of the energy, and that includes any estimate of the temperature.”

J.Uffink, J.van Lith: Thermodynamic Uncertainty Relations Again: A Reply to Lavenda; Found.Phys.Lett.14(2001)187

"In this interpretation, the uncertainty $\Delta\beta$ merely reflects one's lack of knowledge about the fixed temperature parameter β. Thus β does not fluctuate."

"Lavenda's book uses these ingredients to derive the uncertainty relation $\Delta\beta \cdot \Delta U \geq 1$. Our paper observes that, on the same basis, one actually obtains a result even stronger than this, namely $\Delta\beta \cdot \Delta U = 1$."

Outline

Outline

Variances of functions of distributed quantities

Let x be distributed with small variance. Consider a Taylor expandable function

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a) + \dots$$

Up to second order the square of it is given by

$$f^2(x) = f^2 + 2(x-a)ff' + (x-a)^2 \left[f'f' + ff''\right] + \dots$$

denoting $f(a)$ shortly by f. Expectation values as integrals deliver

$$\langle f \rangle = f + \frac{1}{2}\Delta x^2 f'' \qquad \langle f \rangle^2 = f^2 + \Delta x^2 ff'' \qquad \langle f^2 \rangle = f^2 + \Delta x^2 (f'f' + ff'')$$

Finally we obtain $\boxed{\Delta f = |f'|\,\Delta x}$

One Variable EoS: $S(E)$

Product of variances

$$\Delta E \cdot \Delta\beta = 1 \tag{1}$$

Connection to the (absolute) temperature:

$$|C|\Delta T \cdot \frac{\Delta T}{T^2} = 1 \tag{2}$$

Relative variance scales like 1/SQRT of heat capacity!

$$\frac{\Delta T}{T} = \frac{\Delta\beta}{\beta} = \frac{1}{\sqrt{|C|}} \tag{3}$$

C is proportional to the heat bath size (volume, number of degrees of freedom) in the thermodynamical limit.

Gauss distributed β values

$$w(\beta) = \frac{1}{\sqrt{2\pi\sigma}}\, e^{-\frac{(\beta - 1/T_0)^2}{2\sigma^2}} \tag{4}$$

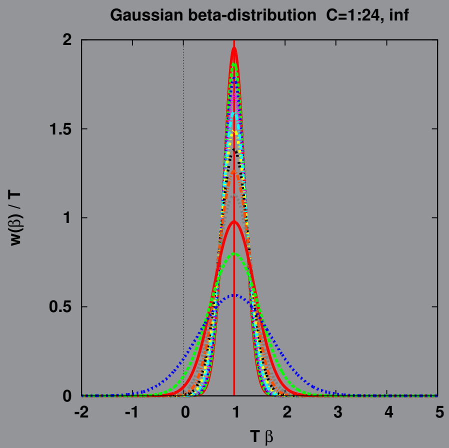
Expectation value

$$\langle \beta \rangle = \frac{1}{T_0}$$

Variance

$$\Delta\beta = \sigma = \frac{1}{T_0\sqrt{|C|}}$$

Plot Gaussian Fluctuations



Superstatistics: one particle energy distribution

Canonical probability for additive thermodynamics:

$$p_i = p(E_i) = e^{\beta(\mu - E_i)}. \tag{5}$$

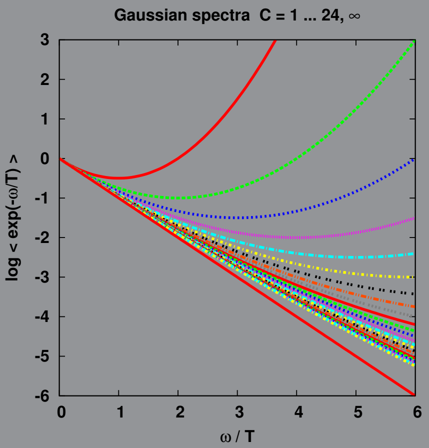
Characteristic function of the Gauss distribution

$$\langle e^{-\beta\omega} \rangle = e^{-\omega/T_0} \, e^{\sigma^2\omega^2/2}. \tag{6}$$

Turning point: maximal energy until when it makes sense

$$E_i^{\max} - \mu = \omega^{\max} = \frac{1}{\sigma^2 T_0} = |C| \, T_0. \tag{7}$$

Plot Gaussian Spectra



J.Uffink, J.van Lith: Thermodynamic Uncertainty Relations; Found.Phys.29(1999)655

"But unlike previous authors, Lindhard considers both the canonical and the microcanonical ensembles as well as intermediate cases, describing a small system in thermal contact with a heat bath of varying size."

"...Lindhard simply assumes that the temperature fluctuations of the total system equal those of its subsystems. This is in marked contrast with all other authors on the subject."

Subsystem - Reservoir System

The "res + sub = tot" splitting interpolates between the

canonical statistics for "sub" $\ll$ "res" $\approx$ "tot" $\longrightarrow \infty$

and the

microcanonical one for "res" $\ll$ "sub" $\approx$ "tot" $\longrightarrow \infty$.

In the simple $S(E)$ analysis, it is $E_{tot} = E_{sub} + E_{res}$

Mutual Info from phase space convolution

$$\Omega(E) = \int dE_1 \, \Omega(E_1) \cdot \Omega(E - E_1). \tag{8}$$

Einstein's postulate: $\Omega(E) = e^{S(E)}$

$$e^{S(E)} = \int dE_1 \, e^{S_1(E_1) + S_2(E - E_1)}. \tag{9}$$

Normalized version:

$$1 = \int dE_1 \, e^{S_1(E_1) + S_2(E - E_1) - S(E)} = \int dE_1 \, e^{I(E, E_1)}. \tag{10}$$

Consider this as a probability distribution for E_1!

Mutual Info in "sub-res" splitting

Mutual information:

$$I(E_{\rm sub}) = S_{\rm sub}(E_{\rm sub}) + S_{\rm res}(E_{\rm tot} - E_{\rm sub}) - S_{\rm tot}(E_{\rm tot}) \tag{11}$$

Let us denote $E_{\rm sub}$ by E in the followings.

$$I'(E) = S'_{\rm sub}(E) - S'_{\rm res}(E_{\rm tot} - E) = \beta_{\rm sub} - \beta_{\rm res}; \tag{12}$$

Saddle point (*zeroth law*):

$$I'(E_*) \; = \; 0 \qquad \Leftrightarrow \qquad \beta_{\rm sub} = \beta_{\rm res} = \frac{1}{T_*} \tag{13}$$

Taylor-expansion of $E - E_*$-fluctuations

for saddle point integrals with factor $e^{I(E)}$:

$$I(E) = I(E_*) + (E - E_*)\, I'(E_*) + \frac{1}{2}\,(E - E_*)^2\, I''(E_*) \tag{14}$$

Gaussian probability: $P(E) = e^{I(E)}$

Second derivative near equilibrium:

$$I''(E_*) = -\frac{1}{T_*^2}\left(\frac{1}{C_{\rm sub}} + \frac{1}{C_{\rm res}}\right) < 0 \tag{15}$$

Small variance approximation

Sub energy fluctuations: $\xi = E - E_*$.
Temperature estimates as $T = 1/\beta$:

$$\langle 1/\beta_{\rm sub} \rangle = \langle 1/\beta_{\rm res} \rangle \approx T_* . \tag{16}$$

Energy and heat capacity are related

$$\langle E \rangle(T) = \int_0^T C(\mathfrak{T}) d\mathfrak{T} \;=\; T \cdot \overline{C}(T) \;\leq\; T \cdot C(T). \tag{17}$$

Gaussian E-variance in equilibrium

Energy expectation value: $\langle E \rangle = \overline{C}_{\rm sub} T_* \le C_{\rm sub} T_*$.

Common temperature: $T_* = \langle 1/\beta_{\rm sub} \rangle = \langle 1/\beta_{\rm res} \rangle$.

Energy variance: $\Delta E_{\rm sub}^2 = \Delta E_{\rm res}^2 = C_* \, T_*^2$

with

$$C_* := \frac{C_{\rm sub} \cdot C_{\rm res}}{C_{\rm sub} + C_{\rm res}} \tag{18}$$

Beta variance: $\Delta \beta_{\rm sub} = \Delta E_{\rm sub} / \left(C_{\rm sub} T_*^2 \right)$.

Products of Gaussian Variances in Equilibrium

$$\Delta\beta_{\rm sub} \cdot \Delta E_{\rm sub} \;=\; \frac{\Delta E_{\rm sub}^2}{C_{\rm sub}\, T_*^2} \;=\; \frac{C_*}{C_{\rm sub}} \;=\; \frac{C_{\rm res}}{C_{\rm sub}+C_{\rm res}} \le 1. \qquad (19)$$

Using the "sub" – "res" symmetry we finally obtain:

$$\Delta\beta_{\rm sub} \cdot \Delta E_{\rm sub} + \Delta\beta_{\rm res} \cdot \Delta E_{\rm res} \;=\; 1. \qquad (20)$$

This generalizes Landau (and many others).

Formulas with Scaled Variances

SUB:

$$\omega^2_{\mathrm{E\,sub}} := \frac{\Delta E^2_{\mathrm{sub}}}{\langle E^2_{\mathrm{sub}} \rangle} \geq \frac{\Delta E^2_{\mathrm{sub}}}{T^2_* C^2_{\mathrm{sub}}} = \frac{C_*}{C^2_{\mathrm{sub}}} \tag{21}$$

RES:

$$\omega^2_{\beta\,\mathrm{res}} := \frac{\Delta \beta^2_{\mathrm{res}}}{\langle \beta^2_{\mathrm{res}} \rangle} = \frac{\Delta E^2_{\mathrm{res}}}{T^2_* C^2_{\mathrm{res}}} = \frac{C_*}{C^2_{\mathrm{res}}} \tag{22}$$

For SUB + RES we finally obtain:

$$C_{\mathrm{sub}}\, \omega^2_{\mathrm{E\,sub}} + C_{\mathrm{res}}\, \omega^2_{\beta\,\mathrm{res}} \geq 1. \tag{23}$$

This resembles Lindhard's (Wilk's,....) formula.

Deficiences of the Gauss picture

1. $w(\beta) > 0$ for $\beta < 0$ (finite probability for negative temperature)
2. $\langle e^{-\beta\omega} \rangle$ is not integrable in ω (it cannot be a canonical one-particle spectrum)

Beyond Gauss: Euler

Euler-Gamma distribution

$$w(\beta) = \frac{a^v}{\Gamma(v)}\, \beta^{v-1}\, e^{-a\beta}. \tag{24}$$

Mean: $\langle\beta\rangle = \frac{v}{a}$, variance: $\frac{\Delta\beta}{\langle\beta\rangle} = \frac{1}{\sqrt{v}}$

Characteristic function

$$\langle e^{-\beta\omega}\rangle = \left(1 + \frac{\omega}{a}\right)^{-v}. \tag{25}$$

Euler adjusted to Gauss

Mean: $\langle\beta\rangle = \frac{v}{a} = \frac{1}{T}$, variance: $\frac{\Delta\beta}{\langle\beta\rangle} = \frac{1}{\sqrt{v}} = \frac{\Delta T}{T} = \frac{1}{\sqrt{|C|}}$

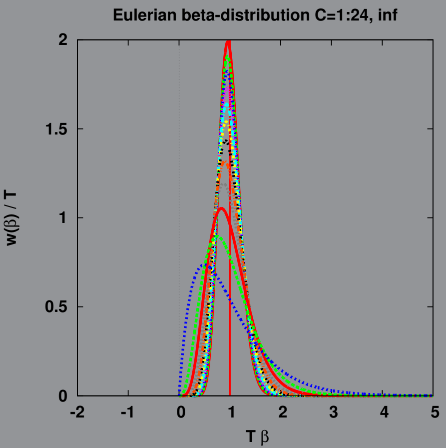
Adjusted Euler-Gamma distribution

$$w(\beta) = \frac{(|C|\,T)^{|C|}}{\Gamma(|C|)}\,\beta^{|C|-1}\,e^{-|C|T\beta}. \tag{26}$$

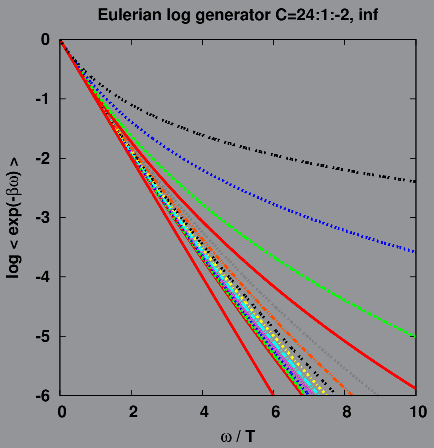
Characteristic function

$$\langle e^{-\beta\omega}\rangle = \left(1 + \frac{\omega}{|C|T}\right)^{-|C|} \underset{|C|\to\infty}{\longrightarrow} e^{-\omega/T}. \tag{27}$$

Plot Eulerian Fluctuations



Plot Eulerian Spectra



Euler for ideal gas

Distribution of the **kinetic energy sum** (non-relativistic):

$$P(E) = \prod_{j=1}^{3N} dp_j \, w(p_j) \, \delta\left(E - \sum_{i=1}^{3N} \frac{p_i^2}{2m} \right). \tag{28}$$

with **Gaussian** (Maxwell-Boltzmann) distribution of the individual p_i components:

$$w(p) = \frac{\sqrt{\beta}}{\sqrt{2\pi m}} \, e^{-\beta \frac{p^2}{2m}} \tag{29}$$

Euler for E and β

Fourier expanding the Dirac-delta we carry out the same integral $3N$ times:

$$P(E)\,dE \;=\; \frac{1}{\Gamma(\frac{3}{2}N)}\,(\beta E)^{\frac{3}{2}N-1}\,e^{-\beta E}\,d(\beta E). \tag{30}$$

This is an Euler-Gamma distribution of β for fix E.

... and a Poissonian for N for fix βE

With $C = 3N/2$ heat capacity, we have $\langle E\rangle = CT$ and

$$\frac{\Delta E}{\langle E\rangle} = \frac{\Delta\beta}{\langle\beta\rangle} = \frac{\Delta T}{\langle T\rangle} = \frac{1}{\sqrt{C}}.$$

Product of Variances for Euler

Since

$$\frac{\Delta E}{\langle E \rangle} = \frac{\Delta \beta}{\langle \beta \rangle} = \frac{1}{\sqrt{C}}$$

and

$$\langle E \rangle \cdot \langle \beta \rangle = C,$$

We derive

$$\frac{\Delta E}{\langle E \rangle} \cdot \frac{\Delta \beta}{\langle \beta \rangle} = \frac{1}{\sqrt{C}} \cdot \frac{1}{\sqrt{C}} = \frac{1}{C}, \qquad \frac{\Delta E \cdot \Delta \beta}{C} = \frac{1}{C}, \tag{31}$$

$$\Delta E \cdot \Delta \beta \; = \; 1$$

Product of Variances for Scaling Fluctuations

For scaling fluct-s, i.e. $P(E) = \beta f(\beta E)$ and $w(\beta) = Ef(E\beta)$ with the *same* function $f(x)$

$$\langle E \rangle = T \int x f(x) dx = CT, \qquad \langle \beta \rangle = \frac{1}{E} \int x f(x) dx = C/E. \tag{32}$$

It is easy to obtain also that

$$\Delta E^2 = T^2 \Delta x^2, \qquad \Delta \beta^2 = \frac{1}{E^2} \Delta x^2. \tag{33}$$

Conslusion:

$$\frac{\Delta E}{\langle E \rangle}_{\text{fix } \beta} = \frac{\Delta \beta}{\langle \beta \rangle}_{\text{fix E}} = \frac{\Delta T}{\langle T \rangle}_{\text{fix E}} = \frac{\Delta x}{\langle x \rangle}$$

Outline

1. Temperature and Energy Fluctuations
2. Finite Heat Bath Effects
   - Generalized Zeroth Law
3. Entropy formulas from zero mutual Information

Thermodynamical Temperature

Biro,Van,PRE83:061147,2011

Thermal exchange of energy $\longrightarrow$ equality of temperature

$$S(E) = max, \qquad \text{while} \quad E = E_1 \oplus E_2, \quad S = S_1(E_1) \oplus S_2(E_2).$$

In general:

$$\begin{aligned} dS &= \frac{\partial S}{\partial S_1} S_1'(E_1) dE_1 + \frac{\partial S}{\partial S_2} S_2'(E_2) dE_2 = 0, \\ dE &= \frac{\partial E}{\partial E_1} dE_1 + \frac{\partial E}{\partial E_2} dE_2 = 0. \end{aligned} \tag{34}$$

Zero determinant solution:

$$\frac{\partial S}{\partial S_1} \frac{\partial E}{\partial E_2} S_1'(E_1) = \frac{\partial S}{\partial S_2} \frac{\partial E}{\partial E_1} S_2'(E_2)$$

Zeroth Law

T.S. Biro, P. Van, Phys.Rev.E 83 (2011) 061147

Zero determinant condition: does it factorize?

Only if:

$$L(E) = L_1(E_1) + L_2(E_2), \qquad K(S) = K_1(S_1) + K_2(S_2).$$

In this case the **thermodynamic temperature** is given by

$$\frac{1}{T} = \frac{\partial K(S)}{\partial L(E)}. \tag{35}$$

Note: such rules are associative and derived as limiting cases for subdividing an arbitrary rule in *T.S.Biro EPL 84 (2008) 56003*

Example: additive composition

This is the leading term for big systems...

$$\begin{aligned} S &= S_1 + S_2, \qquad & K(S) &= S \\ E &= E_1 + E_2, \qquad & L(E) &= E \end{aligned} \tag{36}$$

T-temperature:

$$\frac{1}{T} = S'(E)$$

Example: slightly non-additive composition

This includes subleading terms...

$$\begin{aligned} S &= S_1 + S_2 + aS_1S_2, \qquad & K(S) &= \frac{1}{a}\ln(1+aS) \\ E &= E_1 + E_2 + bE_1E_2, \qquad & L(E) &= \frac{1}{b}\ln(1+bE) \end{aligned} \tag{37}$$

T-temperature:

$$\frac{1}{T} = \frac{1+bE}{1+aS} S'(E)$$

Example: ideal gas   T.S. Biro, Physica A 392 (2013) 3132

Equation of state derivation for E-independent C:

$$S'(E) = \frac{1}{T}, \qquad S''(E) = -\frac{1}{CT^2} \tag{38}$$

Integrations

$$\begin{aligned} -\frac{S''(E)}{S'(E)^2} &= \frac{1}{C}, \\ \frac{1}{S'(E)} &= \frac{E}{C} + T_0, \\ S(E) &= C \ln\left(1 + \frac{E}{CT_0}\right), \end{aligned} \tag{39}$$

with $S(0) = 0$.

Example: ideal gas T.S. Biro, Physica A 392 (2013) 3132

There is **mutual information**:

$$\begin{aligned} I &= S(E_1) + S(E_2) - S(E) \\ &= C_1 \ln\left(1 + \frac{E_1}{C_1 T_0}\right) + C_2 \ln\left(1 + \frac{E_2}{C_2 T_0}\right) - C \ln\left(1 + \frac{E}{C T_0}\right) \end{aligned} \tag{40}$$

Superstatistical and Rényi interpretation with

$$-\ln p_i = -\ln\langle e^{-\beta E_i}\rangle = C_i \ln\left(1 + \frac{E_i}{C_i T_0}\right) = S(E_i)$$

we obtain

$$I = \ln \frac{p}{p_1 p_2}. \tag{41}$$

Outline

Example: ideal gas   T.S. Biro, Physica A 392 (2013) 3132

$I_S \neq 0$ for additive energy $E = E_1 + E_2$.

What is another entropy, $K(S)$, for $I_{K(S)} = 0$ (i.e. **K(S) additive**)?

Answer: $K(S) = \lambda E + \mu$; with $K(0) = 0$ and $K'(0) = 1$ the unique formula is

$$K(S) = C\left(e^{S/C} - 1\right) \tag{42}$$

Example: ideal gas T.S. Biro, Physica A 392 (2013) 3132

Several subsystems: $K_N(S) = \sum_i K_i(S_i)$

Several repeated subsystems $\sim$ **ensemble**:
$K_{\sum N_i}(S) = \sum_i N_i K_i(S_i)$.

Probability interpretation: $N_i = Np_i, \quad N = \sum_i N_i; \qquad \sum_i p_i = 1$

$$K(S) = \sum_i p_i \, K_i(-\ln p_i) \tag{43}$$

This generalizes the Boltzmann-Gibbs-Planck-Shannon formula.

Entropy formulas $q = 1 - 1/C; \qquad C = 1/(1-q)$

Entropy formula for S additive: **Gibbs**

$$S = \sum p_i \left(-\ln p_i \right), \tag{44}$$

Entropy formula for $K(S)$ additive: **Tsallis**

$$K(S) = \sum p_i \, K(-\ln p_i) = \frac{1}{1-q} \sum \left(p_i^q - p_i \right), \tag{45}$$

Alternative entropy formula: **Rényi**

$$S = K^{-1} \left(K(S) \right) = \frac{1}{1-q} \ln \sum p_i^q . \tag{46}$$

Composition rule

T.S. Biro, Physica A 392 (2013) 3132

Formally additive (*formal logarithm*): $K(S) = C\left(e^{S/C} - 1\right)$

Composition rule

$$S = C \ln\left(e^{S_1/C} + e^{S_2/C} - 1\right) = S_1 + S_2 - \frac{1}{C} S_1 S_2 + \dots \quad (47)$$

For $K(0) = 0$ and $K'(0) = 1$ the subleading rule is always Rényi-Tsallis-like.

Generalization T.S.Biró, P.Ván, G.G.Barnaföldi, EPJA 49: 110, 2013

Additive E, $K(S)$ (non-additive S)

Maximal q-entropy of two systems:

$$K(S(E_1)) + K(S(E - E_1)) = \max. \tag{48}$$

First derivative wrsp E_1 is zero $\Longrightarrow$

$$K'(S(E_1)) \cdot S'(E_1) = K'(S(E - E_1)) \cdot S'(E - E_1) = \beta_K \tag{49}$$

Traditional *canonical* approach: $E_1 \ll E$.

Generalization T.S.Biró, P.Ván, G.G.Barnaföldi, EPJA 49: 110, 2013

$$S(E - E_1) = S(E) - S'(E)\,E_1 + \ldots$$

Effects to higher order in E_1/E are better compensated in the following expression

$$\beta_K = K'(S(E))\cdot S'(E) - \left[S'(E)^2\,K''(S(E)) \,+\, S''(E)\,K'(S(E))\right] E_1 + \ldots$$

if the square bracket vanishes.

This we call **Universal Thermostat Independence** - UTI - principle.

Generalization T.S.Biró, P.Ván, G.G.Barnaföldi, EPJA 49: 110, 2013

This leads to the **UTI equation**:

$$\frac{K''(S)}{K'(S)} = -\frac{S''(E)}{S'(E)^2} = \frac{1}{C(S)}. \tag{50}$$

for a general eos leading to an arbitrary $C(S)$ relation.

At the same time the thermodynamical temperature no more coincides with the spectral temperature:

$$\frac{1}{T} = K'(S(E)) \cdot S'(E) = \frac{\partial K(S(E))}{\partial E} = \frac{1}{T_{\mathrm{Gibbs}}} \cdot K'(S). \tag{51}$$

Example: ideal radiation

$$E = \sigma V T^4, \qquad pV = \frac{1}{3}\sigma V T^4, \qquad S = \frac{4}{3}\sigma V T^3.$$

Heat capacities:

$$C_V = 4\sigma V T^3 = 3S, \qquad C_S = \sigma V T^3 = \frac{3}{4}S, \qquad C_p = \infty \tag{52}$$

$K(S)$-formula for th e.o.s. class $C = S/(b-1)$:

$$K(S) = \frac{K'(S_0)}{b}\left[\sum_i p_i \left(\frac{-\ln p_i}{S_0}\right)^b - 1\right] + K(S_0). \tag{53}$$

Summary

- There are temperature fluctuations, they cannot be Gaussian.
- Ideal gas suggests Euler-Gamma distribution and q-entropy formulas.
- UTI principle generalizes the entropy formula construction procedure.

- Outlook
  - Need for realistic modelling of the finite heat bath in heph.
  - Adiabatically expanding systems differ from constant volume systems.
  - Non-extensivity must mean a finite C for infinite V or N.
  - Is there a Minimal Mutual Information Principle?

BACKUP SLIDES

Ideal Gas: Thermodynamics

Constant Heat Capacity $-\frac{S''}{(S')^2} = \frac{1}{C}.$

Integrals: temperature and entropy

$$T = \frac{1}{S'} = \frac{E}{C} + T_0, \qquad S = C \ln\left(1 + \frac{E}{CT_0}\right) + S_0.$$

Mutual info based probability

$$\mathfrak{P}(E_1) = e^{I(E_1)} \propto \left(1 + \frac{E_1}{C_1 T_0}\right)^{C_1} \left(1 + \frac{E - E_1}{C_2 T_0}\right)^{C_2}$$

Ideal Gas: max prob point

$$I'(E_1^*) = \frac{1}{T_0 + E_1^*/C_1} - \frac{1}{T_0 + (E - E_1^*)/C_2} = 0.$$

Common temperature

$$T_* = T_0 + E_1^*/C_1 = T_0 + (E - E_1^*)/C_2$$

Energy sharing

$$\left(\frac{1}{C_1} + \frac{1}{C_2}\right) E_1^* = \frac{1}{C_2} E; \qquad E_1^* = \frac{C_*}{C_2} E = \frac{C_1}{C_1 + C_2} E.$$

Ideal Gas: temperature and its estimators

Since

$$1 + \frac{E_1}{C_1 T_0} = \frac{T_1}{T_0}; \qquad 1 + \frac{E - E_1}{C_2 T_0} = \frac{T_2}{T_0}$$

the temperature (estimator) distribution becomes

$$\mathfrak{P} \propto \left(\frac{T_1}{T_0}\right)^{C_1} \left(\frac{T_2}{T_0}\right)^{C_2}$$

Ideal Gas: removing T_0

Express the actual estimator temperatures via the common T_*:

$$T_1 = T_0 + \frac{E_1}{C_1} = T_0 + \frac{E_1^*}{C_1} + \frac{E_1 - E_1^*}{C_2} = T_* + \frac{E_1 - E_1^*}{C_1}$$

$$T_2 = T_* - \frac{E_1 - E_1^*}{C_2}.$$

Difference fluctuates, weighted sum is fixed!

$$T_1 - T_2 = (E_1 - E_1^*)\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{E_1 - E_1^*}{C_*}$$

$$C_1\,T_1 + C_2\,T_2 = (C_1 + C_2)\,T_*$$

Ideal Gas: temperature distribution

It is an **Euler-Beta distribution**

$$\mathfrak{P}(T_1) \propto T_1^{C_1} \left(T_* - \frac{C_1}{C_2}(T_1 - T_*) \right)^{C_2}$$

in the scaling variable: $x = C_1 T_1/(C_1 + C_2)T_* = C_* T_1/C_2 T_*$

$$\mathfrak{B}(x) = \frac{\Gamma(C_1 + C_2 + 2)}{\Gamma(C_1 + 1)\Gamma(C_2 + 1)} x^{C_1} (1 - x)^{C_2}$$

Beta distribution in x, binomial in C_1 at fix $C_1 + C_2$, etc.

Ideal Gas: limits

- Huge reservoir ($C_2 \to \infty$: for $x = C_1 T_1 / T_*$)
  Euler-Gamma

$$\lim_{C_2 \to \infty} \mathfrak{B}(x) = \frac{1}{\Gamma(C_1 + 1)} x^{C_1} e^{-x}.$$

Ideal Photon Gas: Basic Quantities

Thermodynamic quantities from parametric Equation of State

$$E = \sigma T^4 V, \qquad pV = \frac{1}{3}\sigma T^4 V$$

Gibbs equation

$$TS = E + pV = \frac{4}{3}\sigma T^4 V$$

Entropy and Photon Number

$$S = \frac{4}{3}\sigma T^3 V, \qquad N = \chi\sigma T^3 V.$$

Ideal Photon Gas: Differentials

$$dE = 4\sigma T^3 V dT + \sigma T^4 dV$$

$$dp = \frac{4}{3}\sigma T^3 \, dT$$

$$dS = 4\sigma T^2 V dT + \frac{4}{3}\sigma T^3 dV$$

$$dN = 3\chi\sigma T^2 V dT + \chi\sigma T^3 dV$$

Ideal Photon Gas: Heat Capacities

BLACK BOX scenario (V=const.)

$$C_V = 4\sigma T^3 V = 3S = 4\chi N, \qquad \left.\frac{\Delta T}{T}\right|_V = \frac{1}{2\sqrt{\chi N}}$$

ADIABATIC EXPANSION scenario (S=const.)

$$C_S = \sigma T^3 V = \frac{1}{4} C_V, \qquad \left.\frac{\Delta T}{T}\right|_S = \frac{1}{\sqrt{\chi N}}$$

IMPOSSIBLE scenario (p=const.)

$$C_p = \infty, \qquad \left.\frac{\Delta T}{T}\right|_p = 0$$

Ideal Photon Gas: Relations between Variances

Always: $$\frac{\Delta S}{S} = \frac{\Delta N}{N}$$

BLACK BOX (V=const.): $$\frac{\Delta V}{V} = 0 \qquad \frac{\Delta N}{N} = 3\,\frac{\Delta T}{T}$$

ADIABATIC (S=const.): $$\frac{\Delta V}{V} = 3\,\frac{\Delta T}{T} \qquad \frac{\Delta N}{N} = 0$$

ENERGETIC (E=const.): $$\frac{\Delta V}{V} = 4\,\frac{\Delta T}{T} \qquad \frac{\Delta N}{N} = 7\,\frac{\Delta T}{T}$$

Volume or temperature fluctuations or both?
Gorenstein,Begun,Wilk,...

Several Variables: $S(E, V, N, \ldots) = S(X_i)$

Second derivative of S wrsp extensive variables X_i constitutes a metric tensor g^{ij}.

It describes the variance $\Delta Y^i \Delta Y^j$ with Y associated intensive variables.

Its inverse tensor g_{ij} comprises the variance squares and mixed products for the X_i-s.

How to measure all this ?

- Fit Euler-Gamma or cut power-law $\Longrightarrow T, C$
- Check whether $\Delta T/T = 1/\sqrt{C}$
- If two different C-s, imply "sub + res" splitting
- Check E and ΔE by multiparticle measurements
- Vary T by $\sqrt{s}$ and C by N_{part}