QCD thermodynamics: beyond perturbation theory

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1. Introduction

2. The $\mu = 0$ equation of state

3. Order of phase transition: phase boundary

4. Position of the Critical EndPoint (CEP)

5. Conclusions
1. Introduction

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Challenges in thermodynamics of strong interaction

MC measurement of QCD pressure at physical point

- **EoS from lattice simulations**
  (Sz. Borsanyi et.al. 2010)

- **crossover at** $T_c = 156$ MeV
  (Y. Aoki et.al. 2006)

- **perturbative QCD**: resummations

- **hadrons**: how to treat (unstable) bound states?

- **what is the order parameter?** (is there at all?)

[Graph showing p/T vs T (MeV) with data points and a best fit line labeled as MC data. The curve crosses through SB.]
Challenges in thermodynamics of strong interaction

**MC measurement of QCD pressure at physical point**

- EoS from lattice simulations
  
  \( \text{(Sz. Borsanyi et.al. 2010)} \)

- crossover at \( T_c = 156 \text{ MeV} \)

**Order of the phase transition**

Columbia plot

- crossover PhT in the physical point

- find the 2nd order boundary near chiral point
Challenges in thermodynamics of strong interaction

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Order of the phase transition

Columbia plot

Phase diagram

- critical line
- critical endpoint (CEP)
- iso $S/N$ curves
- color superconductor phase

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Perturbative description of the EoS

(J.O. Andersen et.al. 2014)

- at high energy scales (high temperature): asymptotic freedom \( \Rightarrow \) perturbative QCD; from \( T \gtrsim 200 - 250 \) MeV
at high energy scales (high temperature): asymptotic freedom $\Rightarrow$ perturbative QCD; from $T \gtrsim 200 - 250$ MeV

at low energy scales (low temperature): bound states are formed (hadrons) which interact “weakly” $\Rightarrow$ perturbative hadron gas (HRG) description; up to $T \lesssim 170$ MeV
What drives the phase transition?

(J.O. Andersen et.al. 2014)

Problem

- $p_{HRG}$ overshoots the real pressure
- $F_{HRG} \lesssim F_{pert} \quad QCD$ hadronic phase is always more stable

(T.S. Biró, A.J. 2014)
What drives the phase transition?

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Lesson

hadronic degrees of freedom must disappear from the system!
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Lesson

hadronic degrees of freedom must disappear from the system!

We must learn, how to treat gas of bound states (cf. H-atom)!
Possible (traditional) explanation:

- quarks exist, but have large self-energies
  \[ m_{q,g} \xrightarrow{T<T_c} \infty \]

  thermal weights \( e^{-\beta m} \ll 1 \): no quarks at \( T < T_c \) (confinement)

- hadrons dissociate at high temperature: no hadrons at \( T > T_c \)
Phase transition regime: quasiparticles ideas

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  \[ m_{q,g} \xrightarrow{T<T_c} \infty \]
  thermal weights \( \sim e^{-\beta m} \ll 1 \): no quarks at \( T < T_c \) (confinement)
- hadrons dissociate at high temperature: no hadrons at \( T > T_c \)
- MC data: no drastic variation in masses
- hadronic states are observable even at \( T \sim 1.5 T_c \)

**hadrons do not disappear at \( T_c \)!**
- If hadrons survive \( T_c \) why do not they dominate the pressure?
Particle behaviour in the phase transition regime at $T \sim 156$ MeV (crossover) phase transition

Observations vs. quasiparticle predictions

$150 \lesssim T \lesssim 250$ MeV: non-quasiparticle regime, changing degrees of freedom

nonperturbative methods are needed to describe this regime
Two particles with the same quantum numbers
same quantum number $\Rightarrow$ only their mass can differ!

What do we observe in a mass spectrometer?

- ideally: 2 thin spectral lines

spectrum of two particles

---

Lesson:
changing width (changing spectrum) $\Rightarrow$ changing # of dof.
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Changing degrees of freedom
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- **Width $\sim$ mass difference**: no measurements can resolve the peak structure!
- **The states become indistinguishable $\Rightarrow$ represent 1 dof**
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Lesson:
changing width (changing spectrum) $\Rightarrow$ changing # of dof!
Assume that we know the spectrum (measurement).

Goal: calculate pressure $P(\varrho)$


Strategy

- represent $\varrho$ with a (quadratic) effective model
- calculate thermodynamics from this theory

energy density $\varepsilon = \frac{1}{2} \text{Tr} e^{-\beta H} T_{00}$, use KMS relation

Scalar field case

$$S = \int \frac{d^4q}{(2\pi)^4} \frac{1}{2} \Phi^*(q)\mathcal{K}(q)\Phi(q)$$

for consistency we need a physical spectrum only!

unitary, causal, Lorentz-invariant, $E$, $\vec{p}$ conserving
Result:

**Pressure as a function of the spectral function**

\[ P = \mp T \int \frac{d^4 q}{(2\pi)^4} \frac{\partial \mathcal{K}}{\partial q_0} \ln \left( 1 \mp e^{-\beta q_0} \right) \varrho(q) \]

- generally nonlinear \( \varrho \) dependence due to \( \mathcal{K} \sim \frac{1}{\varrho} \)
  \[ \Rightarrow P \] does not depend on the overall normalization of \( \varrho \).

- for free gas mixture \( \varrho(p) = \sum_i Z_i \delta(p_0 - E_p) \)
  we obtain \( P = \sum_i P^{(0)}(m_i) \): sum of partial pressures; no dependence on \( Z_i \), while they are nonzero!

- cf. Mott-effect, Dashen-formula

Changing degrees of freedom for two particles

How thermodynamics changes when peaks are merged?

- spectrum for two particles with different width, and a typical multiparticle continuum (non-quasiparticle system)
Changing degrees of freedom for two particles

How thermodynamics changes when peaks are merged?

- spectrum for two particles with different width, and a typical multiparticle continuum (non-quasiparticle system)
- at small width $\Rightarrow$ two-particle energy density
- at large width $\Rightarrow$ $\sim$ one-particle energy density
- continuum: practically negligible energy density contribution
Gibbs paradox (actualized)

- In mixture of two bosonic gases the SB limit is $P = N_{\text{eff}} P_{SB}$, where $N_{\text{eff}} = 2$ if the masses are different and $N_{\text{eff}} = 1$ if the masses are equal.

  ⇒ discontinuous change for $\Delta m \to 0$!
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  \[ \Rightarrow \text{discontinuous change for } \Delta m \rightarrow 0! \]
- Gas of free particles explains either \( N_{\text{eff}} = 2 \) or \( N_{\text{eff}} = 1 \)
  no tool to describe the change in \( N_{\text{eff}} \)
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- With changing spectral functions \( N_{\text{eff}} \) is dynamical variable
  \[ \Rightarrow \text{in interacting theories Gibbs paradox is smeared out} \]
Gibbs paradox for interacting gases

Gibbs paradox (actualized)

in mixture of two bosonic gases the SB limit is

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Merging with continuum: melting

- one peak dominated regime: $N_{\text{eff}} = 1$
- continuum dominated regime: $N_{\text{eff}} = 0$
- if peak merges into a continuum $\Rightarrow$ vanishing pressure
- particle ceases to be a thermodynamical dof

thermodynamic definition of $\#$ dof: $N_{\text{eff}}(T) = \frac{P(T)}{P_0(T)}$

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good fitting function: $N_{\text{eff}} = N_0 + N_1 e^{-a\gamma^b}$ (typically $b = 1.5 - 2$)
An oversimplified (statistical) realization of these ideas for QCD

\[ P_{\text{hadr}}(T) = N_{\text{eff}}^{(\text{hadr})} \sum_{n \in \text{hadrons}} P_0(T, m_n), \quad \ln N_{\text{eff}}^{(\text{hadr})} = -(T / T_0)^b, \]

\[ P_{\text{QGP}}(T) = N_{\text{eff}}^{(\text{part})} \sum_{n \in \text{partons}} P_0(T, m_n), \quad \ln N_{\text{eff}}^{(\text{part})} = G_0 - c(N_{\text{eff}}^{(\text{hadr})})^d. \]

\[ P = P_{\text{hadr}} + P_{\text{QGP}} \] total pressure, \( P_0 \) ideal gas pressure

- **hadrons**: Hagedorn-sp. up to a certain mass (\( m \lesssim 3 \text{ GeV} \))
- **partons** quark and gluon quasiparticles
- \( N_{\text{hadr}}(\gamma) \) common suppression factor for all hadrons: stretched exponential, and \( \gamma \sim T \)
- \( N_{\text{part}}(N_{\text{hadr}}) \) partonic suppression factor grows with the \# of available hadronic resonances.
• fit to MC data  Sz. Borsanyi et.al., JHEP 1011 (2010) 077
fit to MC data  \cite{Sz. Borsanyi et al., JHEP 1011 (2010) 077}

- $T < 150 \text{ MeV}$ from HRG using Hagedorn spectrum
  (pion mass input)
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• $T < 150 \text{MeV}$ from HRG using Hagedorn spectrum (pion mass input)
• $\gamma_{\text{hadr}} = \frac{T}{T_0}$, $N_{\text{hadr}} \sim e^{-\gamma^b_{\text{hadr}}}$: fit to avoid large hadron pressure
fit to MC data  

$T < 150 \text{ MeV}$ from HRG using Hagedorn spectrum  
(pion mass input)

$\gamma_{\text{hadr}} = \frac{T}{T_0}, \ N_{\text{hadr}} \sim e^{-\gamma_{\text{hadr}}}$: fit to avoid large hadron pressure

from pressure at $T > 300 \text{ MeV}$ fit QGP parameters  
(fixed $m_q = 330 \text{ MeV}, \ m_g = 600 \text{ MeV}$)
Fit to MC data: Sz. Borsanyi et al., JHEP 1011 (2010) 077

- $T < 150$ MeV from HRG using Hagedorn spectrum (pion mass input)
- $\gamma_{hadr} = \frac{T}{T_0}$, $N_{hadr} \sim e^{-\gamma_{hadr}}$: fit to avoid large hadron pressure

- from pressure at $T > 300$ MeV fit QGP parameters (fixed $m_q = 330$ MeV, $m_g = 600$ MeV)

- quark and gluon width depends on the number of hadrons
  $\gamma_{QGP}^2 = \gamma_0^2 + c N_{hadr}^\alpha$, $N_{QGP} = e^{-\gamma_{QGP}^2}$. 
Introduction

The $\mu = 0$ equation of state

Order of phase transition: phase boundary

Position of the Critical EndPoint (CEP)

Conclusions
• **Physical point:** crossover phase transition ✓

• **chiral limit** (chiral phase transition) and pure Yang-Mills

• tricritical point at \( m_{u,d} = 0 \), chiral line \( m_s \sim m_{u,d}^{2/5} \)

• must be a borderline between the two: where is it?
The chiral critical line

MC measurements and perturbative calculations:

![Graph showing MC measurements and perturbative calculations.](image)

- coarse lattice simulations show borderline
- pert. theory must assume spectrum when not at physical point

The chiral critical line

MC measurements and perturbative calculations:

- coarse lattice simulations show borderline
- pert. theory must assume spectrum when not at physical point
- recent lattice studies suggest much smaller values!


rescaling masses from physical point $m_\pi < 20 - 45$ MeV

The chiral critical line

MC measurements and perturbative calculations:

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- pert. theory must assume spectrum when not at physical point
- recent lattice studies suggest much smaller values!


rescaling masses from physical point $m_\pi < 20 - 45$ MeV
The curvature of the chiral line at small $\mu$

2nd order phase transition surface in $m_{ud} - m_s - \mu$ space

\[ \begin{align*} 
\mu & \quad m_{ud} \quad m_s \quad \mu \\
\end{align*} \]


- $m_{ud} = m_s = m_c$ (diagonal) line: $m_c(\mu) < m_c(0)$ from coarse lattice (P. de Forcrand, O. Philipsen, 2007,2008) $\Rightarrow$ no CEP?

- perturbative calculation: bends toward the physical point (P. Kovacs, Zs. Szep, PRD 75 (2007) 025015)

- RG study: perturbative result very sensitive to details (AJ, Zs. Szep, PRD 82 (2010) 125038)
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M.A. Stephanov (2007): **almost** all available results
(M.A. Stephanov, PoS LAT2006:024,2006, [hep-lat/0701002])

- **black points:** pert. calculations, **green diamonds:** lattice, **red circles:** heavy ion freezout
- **all results depends strongly on the applied method!!**
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- **black points**: pert. calculations, **green diamonds**: lattice, **red circles**: heavy ion freezout
- **all results depends strongly on the applied method!!**
- **What is the present status?**
Perturbative studies

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- black points: pert. calculations, green diamonds: lattice, red circles: heavy ion freezout
- all results depends strongly on the applied method!!
- What is the present status?
- Why are there so large differences? What are the important effects?
MC calculation of phase diagram

Formula to compute on lattice:

\[
\text{Tr } e^{-\beta(H-\mu N)} \hat{O} = \frac{1}{Z} \int \mathcal{D}U \ e^{-S_g(\det \mathcal{M}(\mu))} \tilde{O}[U]
\]

where

- \(S_g\) gauge action
- \(S_f(\mu) = \int d^4x \bar{\Psi} (D_\mu \gamma_\mu + m - \gamma_0 \mu) \Psi = \int d^4x \bar{\Psi} \mathcal{M}(\mu) \Psi\)
  fermionic action with chemical potential

**Problem (sign problem):** \(\det \mathcal{M}(\mu)\) is not real!

\[
\gamma_5 \mathcal{M}(-\mu) \gamma_5 = \mathcal{M}^\dagger(\mu) \quad \Rightarrow \quad \det \mathcal{M}^*(\mu) = \det \mathcal{M}(-\mu).
\]

**Consequence:** \(e^{-S_g[U]} \det \mathcal{M}(\mu)\) is not a probability measure

\(\Rightarrow\) **No importance sampling!**
Idea: generate configurations at $T' \neq T$ and $\mu = 0$, and use them to calculate the finite $\mu$ case:

\[
\text{Tr } e^{-\beta(H-\mu N)} = \int \mathcal{D}U \, e^{-S_g(\beta)}(\det \mathcal{M}(\beta, \mu)) =
\]

\[
= \int \mathcal{D}U \frac{e^{-S_g(\beta)}(\det \mathcal{M}(\beta, \mu))}{e^{-S_g(\beta')}(\det \mathcal{M}(\beta', 0))} e^{-S_g(\beta')}(\det \mathcal{M}(\beta', 0)) =
\]

\[
= \left\langle \frac{e^{-S_g(\beta)}(\det \mathcal{M}(\beta, \mu))}{e^{-S_g(\beta')}(\det \mathcal{M}(\beta', 0))} \right\rangle Z(\beta', \mu = 0).
\]

The phase diagram

Critical endpoint


\[T_E = 162 \pm 2 \text{ MeV}, \]
\[\mu_E = 360 \pm 40 \text{ MeV}.\]
How reliable is this result?

Radius of convergence of rescaling (de Forcrand 2010)

- rescaling: ratio of two partition functions with energy difference
  \[
  \frac{Z(\mu)}{Z(0)} = e^{-\beta V \Delta f}
  \]

  ⇒ overlap exponentially vanishes for large volumes
  Statistics grows with $\sqrt{N_{\text{step}}}$ ⇒ exponentially large
  number of steps are required in the thermodynamic limit

- characterization of sign problem: isospin chemical potential $\mu$
  \[
  \langle e^{2i\Theta} \rangle = \left\langle \frac{\det^2 M(\mu)}{|\det M(\mu)|^2} \right\rangle \sim e^{-\#\mu^2} \quad \Rightarrow \quad \# \text{ config.} \sim e^{\#\mu^2}.
  \]

- At imaginary $\mu/T \approx i\pi/3$ Roberge-Weiss phase transition
  ⇒ restrict radius of convergence

Conclusion

MC methods are reliable for $\mu_B^3 = \mu < T$
How reliable is this result?

Radius of convergence of rescaling (de Forcrand 2010)

- rescaling: ratio of two partition functions with energy difference
  \[ \frac{Z(\mu)}{Z(0)} = e^{-\beta V \Delta f} \]

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  Statistics grows with \( \sqrt{N_{\text{step}}} \)  \( \Rightarrow \) exponentially large
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- characterization of sign problem: isospin chemical potential \( \mu \)
  \[ \langle e^{2i\Theta} \rangle = \left\langle \frac{\det^2 M(\mu)}{|\det M(\mu)|^2} \right\rangle \sim e^{-\#\mu^2} \Rightarrow \# \text{ config.} \sim e^{\#\mu^2}. \]

- At imaginary \( \mu/T \approx i\pi/3 \) Roberge-Weiss phase transition
  \[ \Rightarrow \text{restrict radius of convergence} \]

**Conclusion**

MC methods are reliable for \( \frac{\mu B}{3} = \mu \lesssim T \)
Latest MC results

(Bellwied et.al (BMW group) 2015)

- Imaginary $\mu_B$ calculation (no sign problem)
- Taylor expand results in $\mu$ and continue to real axis

![Graph showing temperature vs. baryonic chemical potential]

- left panel: radius of convergence
- right panel: for all $\mu_B$ the susceptibility curves can be scaled to each other $\Rightarrow$ analytic

$\mu_B = 0.00 \text{ T}$
$\mu_B = 1.18 \text{ T}$
$\mu_B = 1.57 \text{ T}$
$\mu_B = 1.96 \text{ T}$
Latest MC results

(Bellwied et.al (BMW group) 2015)

- Imaginary $\mu_B$ calculation (no sign problem)
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Consequence

no sign of a phase transition until $\mu_B \lesssim 400$ MeV!
Semianalytic method

(cf. P. Kovács dissertation)

- CEP found at nonphysical pion mass, not at cont. limit.
- **one-parameter scaling hypothesis**: assume that one parameter determines the extrapolation to physical point
  choose $\Delta T(\chi)$ width of the susceptibility curve

\[ \mu_{CEP} \text{ vs } \Delta T(\chi, \mu = 0): \text{ model calculations fits to numerical data} \]

(Fodor, Katz 2001, 2004).

- Phyiscal point at $\mu = 0$: width of susc. curve is $\Delta T(\chi) \approx 28$ MeV

(Aoki, Fodor et.al. 2006)

**Prediction:** $\mu_{CEP} \sim 1000$ MeV
Effective model: **chiral sigma model**

\[
\mathcal{L} = \frac{1}{2} \varphi (-d^2 - m^2) \varphi - \frac{\lambda}{24N} (\varphi^2)^2 + \bar{\psi} [i \gamma_5 \partial - m_q] \psi - \frac{g}{\sqrt{N}} \varphi T \psi
\]

(\varphi = (\sigma, \pi_a), T = (1, i\sqrt{2N_f} T_a \gamma_5), N = 4, N_f = 2)

- 1-loop resummed perturbation theory in large \(N\) expansion
- effective potential (free energy) \(\Rightarrow\) phase transition at

\[
\frac{g^2 N_f}{2\pi^2} \mu^2 + \left( \frac{\lambda}{36} + \frac{g^2 N_f}{6} \right) T^2 = m^2_\sigma
\]

\(\Rightarrow\) an ellipse in the \(\mu - T\) plane

- position of the CEP analytically determined
Effective model: chiral sigma model

\[ \mathcal{L} = \frac{1}{2} \varphi(-d^2 - m^2) \]

\( \varphi = (\sigma, \pi_a), \quad T = (1, i\sqrt{2}) \)

- 1-loop resummed perturbation theory in large \( N \) expansion
- Effective potential (free energy): \( \Rightarrow \) phase transition at
  \[ g^2 N_f \frac{\lambda^3}{2\pi^2} \mu^2 \]
  \( \Rightarrow \) an ellipse in \( T, \mu \) plane
- position of the CEP analytically determined

Prediction: \( \mu_{\text{CEP}} \sim 850 \text{ MeV} \)
Bosonic fluctuation

$(\lambda, g)$ space: either first order or second order phase transition; border line tricritical points.

- Mean field predicts strongest phase transition $\exists (\lambda, g)$ where mean field predicts 1st order, FRG 2nd order PhT.
- Bosonic fluctuations soften the transition
- One-loop is quite good

To mimic \textit{bosonic fluctuations} in the gauge sector, the effect of \textit{confinement} (bound states) one may introduce the Polyakov-loop.


$$\Phi = \frac{1}{N_c} \left< \text{Tr} \mathcal{P} e^{\int_0^\beta i g \, d\tau A_0} \right>$$

- educated guesses to the form of the Polyakov-loop potential, fitted to Yang-Mills and QCD thermodynamics; eg.:


$$\frac{U(\Phi, \bar{\Phi})}{T^4} = -\frac{b_2(T)}{2} \Phi \Phi - \frac{b_3}{6} (\Phi^3 + \bar{\Phi}^3) + \frac{b_4}{4} (\Phi \Phi)^2.$$  

- Polyakov-loop potential treated in mean field

- confined phase: $\Phi = 0$ ($Z_3$ symmetric phase)
  deconfined phase: $\Phi \neq 0$ ($Z_3$ broken phase)

- $\Phi \Rightarrow$ modified Fermi-Dirac distribution functions

$$N_q(T, \mu; \bar{\Phi}, \Phi) = \frac{1 + 2 \Phi e^{(E_q - \mu)/T} + \Phi e^{2(E_q - \mu)/T}}{1 + 3 \bar{\Phi} e^{(E_q - \mu)/T} + 3 \Phi e^{(E_q - \mu)/T} + e^{3(E_q - \mu)/T} \Phi \Phi}.$$
Other analytic models, role of bosonic fluctuations

QM large N with bosonic fluctuations


QM: quark-meson, P: Polyakov-loop, DSE: Dyson-Schwinger-eq.
Other analytic models, role of bosonic fluctuations

chiral sigma model with/without Polyakov loops

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Other analytic models, role of bosonic fluctuations

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Other analytic models, role of bosonic fluctuations

PQM with FRG (full fluctuations)

QM: quark-meson, P: Polyakov-loop, DSE: Dyson-Schwinger-eq.
Other analytic models, role of bosonic fluctuations

Vector meson extended PQM

QM: quark-meson, P: Polyakov-loop, DSE: Dyson-Schwinger-eq.

(P. Kovács, Zs. Szép, Gy. Wolf, 2016)
Other analytic models, role of bosonic fluctuations

Some relevant results

**Lesson**

- correct treatment of bosonic fluctuations are important!
- analytic methods with fluctuations: $\mu_{B,CEP} \approx 800 - 1000 \text{ MeV}$
- DSE pure QCD approach: $\mu_{B,CEP} \approx 500 \text{ MeV}$ (better fermionic fluctuations?)
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Conclusions

- Thermodynamics of strongly interacting matter is perturbative for $T < 150$ MeV (HRG), and $T > 250$ MeV (QCD) (at $\mu = 0$)
- in the critical domain (analytically) changing dof
  $\Rightarrow$ hadron melting
  **crucial:** correct treatment of spectral properties
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Columbia plot: borderline between 1st order phtr. and crossover: not seen in MC

Does the critical surface bend at small $\mu$ toward or away from the physical point?
Conclusions

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- Columbia plot: borderline between 1st order phtr. and crossover: not seen in MC
- Does the critical surface bend at small $\mu$ toward or away from the physical point?
- location of the CEP: direct MC methods $\mu_{B,CEP} > 450$ MeV
  - crucial: correct treatment of bosonic fluctuations (direct or Polyakov loops)
- latest results: $\mu_{B,CEP} \approx 800 - 1000$ MeV
Thermodynamics from spectral function II.

We start from the Lagrangian:
\[ \mathcal{L} = \frac{1}{2} \Phi^*(q) \mathcal{K}(q) \Phi(q) \]

- In order to reproduce the given \( \varrho \) spectral function we need
  \[ \varrho = \text{Disc} i \mathcal{K}^{-1}, \quad \mathcal{K}^{-1}(q) = \int \frac{d\omega}{2\pi} \frac{\varrho(\omega, q)}{q_0 - \omega} \]

- Energy momentum tensor (Noether-current):
  \[ T_{\mu\nu}(x) = \frac{1}{2} \varphi(x) D_{\mu\nu} \mathcal{K}(i\partial) \varphi(x) \]

where
\[ D_{\mu\nu} \mathcal{K}(i\partial) = \left[ \frac{\partial \mathcal{K}(p)}{\partial p^\mu} p_\nu - g_{\mu\nu} \mathcal{K}(p) \right] \]

and the symmetrized derivative is defined as
\[ f(x)[(i\partial)^n]_{\text{sym}} g(x) = \frac{1}{n+1} \sum_{a=0}^n [(-i\partial)^a f(x)][(i\partial)^{n-a} g(x)]. \]

- We take its expectation value using KMS relation
  \[ \langle \varphi \varphi \rangle(q) = n_{BE}(q_0) \varrho(q) \]

\[ \Rightarrow \text{symmetrized derivative becomes normal one.} \]
Quantum statistical averages can be computed as

\[ \text{Tr} e^{-\beta H} \hat{O} = \int DUD\bar{\Psi}D\Psi \ e^{-S} \ O[\Psi, U], \]

the action consists of a fermion and a gauge part \( S = S_f + S_g \).

- The fermionic part (with \( D \) covariant derivative):

  \[ S_f = \int d^4x \bar{\Psi}(D_\mu \gamma_\mu + m)\Psi = \int d^4x \bar{\Psi} M\Psi, \quad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \]

- The fermionic path integral yields

  \[ \int D\bar{\Psi}D\Psi \ e^{-S_f} = \det M \]

- This contributes to the gauge action as

  \[ \text{Tr} e^{-\beta H} \hat{O} = \int DU \ e^{-S_g}(\det M) \ O[U] \]

Consistency: real expression, since \( (\det \gamma_5 = 1) \)

\[ \gamma_5 M \gamma_5 = M^\dagger \quad \Rightarrow \quad \det \gamma_5 M \gamma_5 = \det M = \det M^\dagger \]

Algorithm: produce configurations with probability \( \sim e^{-S_g + \ln \det M} \)
For a conserved quantity $N_q = \int d^4x \bar{\Psi}_q \gamma_0 \Psi_q$ we can introduce a chemical potential

$$e^{-\beta H} \rightarrow e^{-\beta(H - \mu N)}$$

This modifies the fermionic action

$$S_f(\mu) = \int d^4x \bar{\Psi}(D_\mu \gamma_\mu + m - \gamma_0 \mu)\psi = \int d^4x \bar{\Psi} M(\mu)\psi.$$ 

Problem (sign problem): $\det M(\mu)$ is not real!

$$\gamma_5 M(-\mu) \gamma_5 = M^\dagger(\mu) \Rightarrow \det M^*(\mu) = \det M(-\mu).$$

Consequence: $e^{-S_g[U]} \det M(\mu)$ is not a probability measure

$\Rightarrow$ No importance sampling!
How reliable is this result?

**Numerical arguments** (de Forcrand 2010)

- rescaling: ratio of two partition functions with energy difference
  \[
  \frac{Z(\mu)}{Z(0)} = e^{-\beta V \Delta f}
  \]
  \(\Rightarrow\) overlap exponentially vanishes for large volumes
  Statistics grows with \(\sqrt{N_{\text{step}}}\) \(\Rightarrow\) exponentially large number of steps are required in the thermodynamic limit

- characterization of sign problem: isospin chemical potential \(\mu\)
  \[
  \langle e^{2i\Theta} \rangle = \left\langle \frac{\det^2 \mathcal{M}(\mu)}{|\det \mathcal{M}(\mu)|^2} \right\rangle
  \]
How reliable is this result?

Numerical arguments (de Forcrand 2010)

- Rescaling: ratio of two partition functions with energy difference
  \[ \frac{Z(\mu)}{Z(0)} = e^{-\beta V \Delta f} \]

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- Characterization of sign problem:
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- rescaling: ratio of two partition functions with energy difference
  \[ \frac{Z(\mu)}{Z(0)} = e^{-\beta V \Delta f} \]

  \[ \Rightarrow \] overlap exponentially vanishes for large volumes
  Statistics grows with \( \sqrt{N_{\text{step}}} \) \( \Rightarrow \) exponentially large
  number of steps are required in the thermodynamic limit

- characterization of sign problem: isospin chemical potential \( \mu \)
  \[ \langle e^{2i\Theta} \rangle = \left\langle \frac{\det^2 \mathcal{M}(\mu)}{|\det \mathcal{M}(\mu)|^2} \right\rangle \sim e^{-\#\mu^2} \quad \Rightarrow \quad \# \text{ config.} \sim e^{\#\mu^2}. \]
How reliable is this result?

**Numerical arguments** (de Forcrand 2010)

- Rescaling: ratio of two partition functions with energy difference
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- Characterization of sign problem: isospin chemical potential \(\mu\)
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  \langle e^{2i\Theta} \rangle = \langle \frac{\det^2 \mathcal{M}(\mu)}{|\det \mathcal{M}(\mu)|^2} \rangle \sim e^{-\#\mu^2} \Rightarrow \# \text{ config.} \sim e^{\#\mu^2}.
  \]

- At imaginary \(\mu/T \approx i\pi/3\) Roberge-Weiss phase transition
  ⇒ radius of convergence of the overlap to \(\mu = 0\) case is of the order \(\mu \sim T\) (ie. \(\mu_B \lesssim T\)).

**Conclusion**

MC methods are reliable for \(\frac{\mu_B}{3} = \mu \lesssim T\)
The MC determined CEP is not at continuum limit, not at physical point (large quark masses) . . .

P. Kovács and Zs. Szép had an elegant line of thought to assess the CEP in the physical point (cf. P. Kovács dissertation)

- Assume that Z. Fodor et.al. found the CEP
- one-parameter scaling hypothesis
  - experience: most sensitive quantity is pion mass $m_\pi \Rightarrow$
- Assume that the value of $m_\pi$ determines the extrapolation to physical point
- in effective model calculation determine $m_\pi$-dependence of
  - width of the susceptibility peak $\Delta T(\chi)$
  - position of the CEP $(\mu_{CEP}, T_{CEP})$
- finally determine the $\Delta T(\chi)$ dependence of the CEP!
Model calculation: $\Delta T(\chi)$ vs $m_\pi$ approx. linear

$\mu_{CEP}$ vs $\Delta T(\chi)$ fits to numerical data (Fodor, Katz 2001, 2004).

At physical point at $\mu = 0$ the width of susceptibility curve is $\Delta T(\chi) \approx 28$ MeV (Aoki, Fodor et.al. 2006)

$\Rightarrow$ Prediction $\mu_{CEP} \sim 1000$ MeV
Simplified realization of these ideas to QCD

\[ P = P_{\text{hadr}} + P_{\text{QGP}} \text{ total pressure, } P_0 \text{ ideal gas pressure} \]

\[
P_{\text{hadr}}(T) = N_{\text{eff}}^{(\text{hadr})} \sum_{n \in \text{hadrons}} N_0(T, m_n), \quad \ln N_{\text{eff}}^{(\text{hadr})} = -(T / T_0)^b,
\]

\[
P_{\text{QGP}}(T) = N_{\text{eff}}^{(\text{part})} \sum_{n \in \text{partons}} N_0(T, m_n), \quad \ln N_{\text{eff}}^{(\text{part})} = G_0 - c(N_{\text{eff}}^{(\text{hadr}})^d.
\]

- **hadrons**: Hagedorn-sp. up to a certain mass \((m \lesssim 3 \text{ GeV})\)
- **partons**: quark and gluon quasiparticles
- \(N_{\text{hadr}}(\gamma)\) common suppression factor for all hadrons: stretched exponential, and \(\gamma \sim T\)
- \(N_{\text{part}}(N_{\text{hadr}})\) partonic suppression factor grows with the \# of available hadronic resonances.
Literature:

- **Z. Fodor and S. Katz papers**
  - (“Lattice determination of the critical point of QCD at finite T and mu”, JHEP 0203 (2002) 014)
  - (“Critical point of QCD at finite T and mu, lattice results for physical quark masses”, JHEP 0404 (2004) 050)

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  - (“Simulating QCD at finite density”, PoS LAT2009 (2009) 010)
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- **Ch. S. Fischer, J. Luecker and Ch. A. Welzbacher**